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ABSTRACT

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LOGICAL ERROR ANALYSIS AND CONSTRUCTION OF TESTS TO DIAGNOSE STUDENT "BUGS" IN ADDITION AND SUBTRACTION OF FRACTIONS

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Abstract

This report illustrates a network of procedures which can be used to solve problems involving the addition and subtraction of fractions. This network, which is based on a skills hierarchy, is used to classify seven levels of student competency. The determination of student competency depends upon the careful construction of error-diagnostic tests. Several examples of student response patterns are used to illustrate a procedure to construct a few selected items for such a test so that it will have both content and construct validity. Similiar examples of student misconceptions and incomplete knowledge are included to illustrate the difficulty/futility in using test scores to assess student performance.

The report includes several lists of projected errors which are either predicted from the nodes of the procedural network or are based on classroom observations of junior high school students. These errors have been classified by the node best representing the misconception or incomplete information. Complete tests which were used to assess student knowledge have been included in the report.

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Introduction

The traditional achievement tests, including criterion-referenced tests are constructed for measuring the outcome of treatments, instruction, or the students' past experience in learning. The items in these tests are usually carefully examined in terms of content validity. Tatsuoka & Tatsuoka (1981) demonstrated in their study with signed number arithmetic that examining the content validity of test items is not enough for constructing the items, especially when tests are used in conjunction with instruction. Erroneous rules resulting from a variety of misconceptions and incomplete knowledge produce aberrant response patterns. As a result, the statistics representing the behaviors of items, test scores cannot be reliable and valid either for assessing the students' performances on the tests or for evaluating the efficiency and quality of the treatments. Even if all items are carefully chosen from a single content domain, the test still requires a thorough examination on construct validity. In other words, the underlying cognitive process used by most students must be carefully studied in order to measure the information originally intended.

The error-diagnostic tests for whole-number subtraction problems (Brown & Burton, 1978) and signed-number arithmetic (Tatsuoka, et al., 1980) have successfully diagnosed hundreds of bugs which should be useful in the improvement of teaching and the design of new instructional materials. However, item construction of an error-diagnosing test is quite different from that of other tests. It requires a careful selection of items so that each item plays an important role of uniquely determining the erroneous rules committed by the student (if there are any).

This report includes several examples which illustrate a procedure of such an item construction for addition and subtraction problems in fractions. The methods described in the report are based on the approach adopted by Jan Fair of Creative Publications (1977) and also by one of the authors, Mary F. Klein, who has had 15 years teaching experience at a local junior high school. Thus, procedural networks

mentioned in the text reflect only the number theoretic approach. Therefore, it might be safe to say that our error-diagnostic tests will be used when the students' prior knowledge results from instruction using the same or similar methods.

The list of projected errors given in the report are either deduced from each node of the procedural network by assuming various misconceptions and incomplete knowledge on the students' part or are based on Klein's observations. Although the description of item-construction procedure is given only for a few examples, a 48-item addition and 42-item subtraction test (Appendix II) was carefully constructed by following the procedures given in the examples. The answers to these problems are expected to provide sufficient information to facilitate diagnosis of all the errors presented in the list.

A systematic and general approach to achieve the goal -- item construction of error-diagnosing tests -- should be explored and investigated as a research topic in the future.

Reliability of "Right" or "Wrong" Scoring

In order to diagnose student errors and to assess student achievement, teachers need more than a single raw score, such as the number of items correct on a test. By themselves, such numbers can be misleading. For example, Birenbaum and Tatsuoka (1980) identified several students who had errors of varying degrees of seriousness even though they had identical scores on quizzes on addition and subtraction of signed numbers. Thus, the single raw test score cannot be used to diagnose either the nature of the errors or the degree of seriousness of the errors.

It is likely that the use of a single raw score from a test covering addition and subtraction of fractions would be just as misleading, if not more so. To illustrate, consider some sample addition problems (Table 1) and subtraction problems (Table 2). Using "bugs" which were consistly applied by junior high school students, responses for three hypothetical students were generated. The "bugs" were chosen to illustrate the futility of using a single raw score to diagnose student errors.

Insert Table 1

On the addition "test," students 1, 2 and 3 would have had the same raw score (total correct). All three understand that a common denominator is needed to add fractions. However, each consistently fails to follow some procedure in the addition process. In each case, the "bug" results in some correctly answered items as well as the incorrectly answered ones showing further the difficulties in using one score in assessing student achievement.

Student 1 uses Method B in which the student combines whole-number parts and the fraction parts separately. (A complete description is given in a following section.) When finding the equivalent fractions he uses the bug a/b + c/d -> d/bd + b/bd in which he fails to multiply the numerator of the original fraction to determine the numerator of the equivalent fraction. Rather, he substitutes the denominator of the other fraction. Student 1 is able to answer correctly problems 1 and 5 since equivalent fractions are not needed. Problems 2 and 4 have correct answers because the omitted multiplications are simply multiplications by one and their omission does not affect the answers.

Student 2 fails to add the whole-number parts in problems involving finding a common denominator. He also uses method B (considering whole-number parts and fraction parts separately). He does not recopy the whole-number parts when he finds the equivalent fractions. As a result, he fails to add the whole-number parts correctly.

Student 3 uses Method A correctly (does not separate a mixed fraction into its whole-number portion and its fraction portion). If the answer is a mixed fraction, however, he exchanges the whole-number and the numerator in the answer. The following illustrates how he would solve problem 6 (Table 1):

$$1\frac{1}{6} = \frac{7}{6}$$

$$\frac{2}{3} = \frac{4}{6}$$

$$\frac{11}{6}$$

$$6\overline{)11}\ 1 \quad \frac{6}{5} \quad \rightarrow 5\frac{1}{6}$$

Table 1

ADDITION TEST

Item	Student Answer		
	*1	*2	*3
$\frac{2}{3} + \frac{1}{3} = \frac{3}{3} = 1$	1	1	1
$\frac{1}{3} + \frac{1}{4} = \frac{7}{12}$	$\frac{7}{12}$ U	$\frac{7}{12}$	$\frac{7}{12}$
$\frac{2}{3} + \frac{5}{6} = 1\frac{1}{2}$	$\frac{1}{2}$ X	$1\frac{1}{2}$	$1\frac{1}{2}$ U
$4\frac{1}{5} + 2\frac{1}{3} = 6\frac{8}{15}$	$6\frac{8}{15}$ U	$\frac{8}{15}$ X	$8\frac{6}{15} = 8\frac{2}{5}$ X
$2\frac{2}{5} + 2\frac{2}{5} = 4\frac{4}{5}$	$4\frac{4}{5}$	$\frac{4}{5}$	$4\frac{4}{5}$ U
$1\frac{1}{6} + \frac{2}{3} = 1\frac{5}{6}$	$\frac{1}{2}$ X	$\frac{5}{6}$ X	$5\frac{1}{6}$ X
Percent Correct	66.66%	66.66%	66.66%

X = incorrect response

U = correct response generated by "buggy" method

Problems 3 and 5 are correct because the quotient and the remainder happen to be the same. The exchange is not noticed.

Insert Table 2

The subtraction bug for student 1 involves "borrowing." When the student "borrows" 1 from the whole-number part, he converts it to a 10 which he then adds to the numerator. For example, problem 1 would be solved as follows:

$$\begin{array}{r} 2\frac{1}{\not{4}\,\frac{2}{5}} \\ -\ 2\frac{3}{5} \\ \hline \frac{9}{5} = 1\frac{4}{5} \end{array}$$

The student is transferring his rule directly from whole number subtraction. This procedure is reinforced in problems involving denominators of ten.

Student 2 incorrectly works problems of the type W - F, W - M, and M - W, where W is a whole-number, F is a fraction, and M is a mixed number. (See the next section for a complete description of problem types.) The student simply subtracts the whole-number parts and copies the fraction part to the answer unchanged. His bug does not differentiate between a problem with a fraction in the minuend and one with a fraction in the subtrahend. The assumption seems to be that because this method works in addition, it will work in subtraction.

Student 3 always subtracts the smaller numerator from the larger and then writes the difference as the numerator of the answer. This bug was identified in the subtraction of signed numbers (Birenbaum and Tatsuoka, 1980).

As we can see from these examples, students can apply several different rules in adding and subtracting fractions and still obtain

Table 2

SUBTRACTION TEST

Item	Student Answer		
	*1	*2	*3
$3\frac{1}{10} + 2\frac{5}{10} = \frac{3}{5}$	$\frac{3}{5}$ U	$1\frac{4}{10}$ X	$\frac{3}{5}$
$3\frac{2}{5} - 2\frac{3}{5} = \frac{4}{5}$	$\frac{9}{5} = 1\frac{4}{5}$ X	$1\frac{1}{5}$ X	$\frac{4}{5}$
$8\frac{4}{6} - 6\frac{1}{3} = 2\frac{2}{6} = 2\frac{1}{3}$	$2\frac{1}{3}$	$2\frac{1}{3}$ U	$2\frac{1}{3}$
$6\frac{1}{5} - 3 = 3\frac{1}{5}$	$3\frac{1}{5}$	$3\frac{1}{5}$	$3\frac{1}{5}$ U
$3 - 2\frac{7}{10} = \frac{3}{10}$	$\frac{3}{10}$ U	$\frac{7}{10}$ X	$\frac{7}{10}$ X
$\frac{5}{6} - \frac{1}{3} = \frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{6} = \frac{1}{2}$	$\frac{1}{2}$
$4\frac{2}{3} - 2\frac{2}{3} = 2$	2	2 U	2
Percent Correct	87.50%	37.50%	87.50%

X = incorrect response
U = correct response generated by "buggy" method

correct results. A single raw score would not be an accurate measure of the student's knowledge.

Logical Analysis of Computational Tasks of Fractions

Problem Types

Problems were classified on the basis of three attributes: notation, complexity and size. Table 3 discusses the abbreviations used to explain these attributes.

Insert Table 3

If we consider only fractions (F), whole numbers (W), and mixed fractions (M), we can identify eight fraction addition problem types:

F+F, F+W, F+M, M+F, M+W, M+M, W+F, W+M

Since addition is commutative, three pairs are identical: F+W and W+F, F+M and M+F, and M+W and W+M. Throughout this report, reference to either one of such a pair is meant to include its partner. Thus, there are five distinct problem types for fraction addition:

$$F+F,\ \frac{F+W}{W+F},\ \frac{F+M}{M+F},\ \frac{M+W}{W+M},\ M+W$$

Problems of all eight types appear on the test. The entire set of eight may be needed to differentiate among various bugs. In such a case, both problem types in an equivalent pair aid in analyzing the student's "buggy rule."

Eight different types of subtraction problems were also identified: F-F, F-M, F-W, W-F, W-M, M-F, M-M, M-W. Since subtraction is not commutative, each of the above types is worked differently and, consequently, must be considered as a different problem type. Care was taken to avoid items which would have a negative answer. Obviously, item types such as F-M and F-W could not be included for proper fractions F.

The Procedural Network

There are several different methods for adding and subtracting fractions. One goal of instruction is to provide the students with sufficient and necessary information so that they can select and apply

Table 3

Attributes of Fraction Problems

Abbreviation	Meaning	Examples
W	Whole Number	5, 0, 17
F	Fraction	$\frac{0}{2}$, $\frac{1}{3}$, $\frac{4}{4}$, $\frac{6}{3}$
PF	Proper Fraction	$\frac{0}{2}$, $\frac{1}{3}$
IF	Improper Fraction	$\frac{4}{4}$, $\frac{6}{6}$
M	Mixed Number	$3\frac{1}{2}$, $4\frac{6}{5}$, $1\frac{3}{3}$, $2\frac{0}{7}$
PM	Proper Mixed Number	$3\frac{1}{2}$, $2\frac{0}{7}$
IM	Improper Mixed Number	$4\frac{6}{5}$, $1\frac{3}{3}$

an appropriate and efficient algorithm for solving a particular problem. The procedural network is a representation of the different algorithms which can be used to solve different types of problems. Figures 1A and 1B are the procedural networks for addition and subtraction respectively.

Insert Figure 1A

Insert Figure 1B

Pentagon shaped cells indicate that the particular step is a complicated procedure which is explained in a chart in the appendix. The number of the chart is written in the pentagon. In addition, several of the charts in the appendix contain other procedural networks outlining alternative methods for performing the same step.

Levels of Competency

Computation with fractions requires knowledge and application of a sequence of skills. Within the sequence of skills, certain subsequences can be arranged hierarchically. One such skills hierarchy for adding (subtracting) fractions is shown in Figure 3.

Insert Figure 3

Using Figure 0 we have identified seven levels of competency which are needed to solve problems involving the addition and subtraction of fractions. These levels are classified and discussed in the following paragraphs.

Level I: The student understands how the numerator, denominator and whole number parts of a fraction are related. The general concept of fraction is understood.

Level II: The student is able to add (subtract) two fractions (F+F) with like denominators.

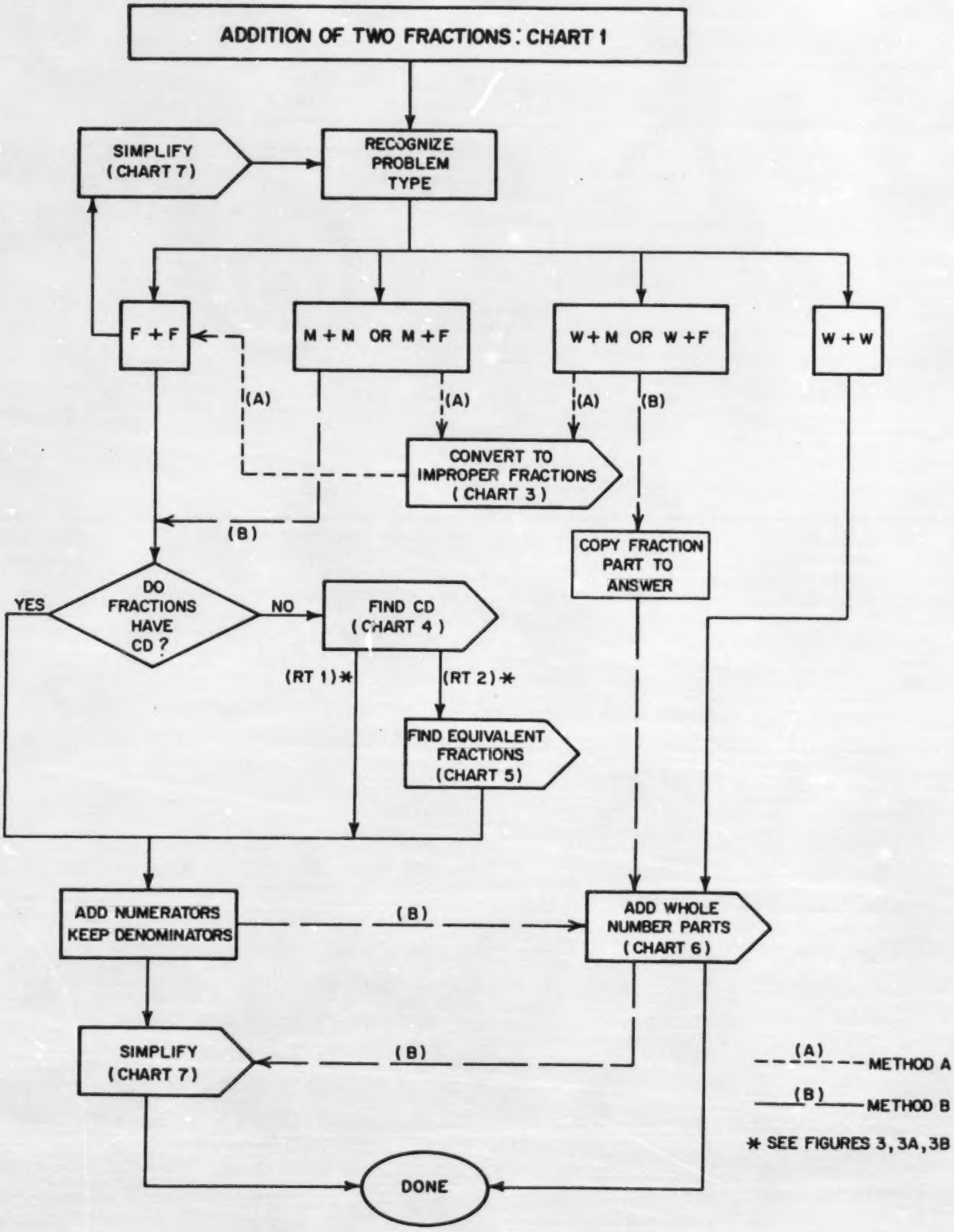

Figure 1A: A Procedural Network for Adding Two Fractions

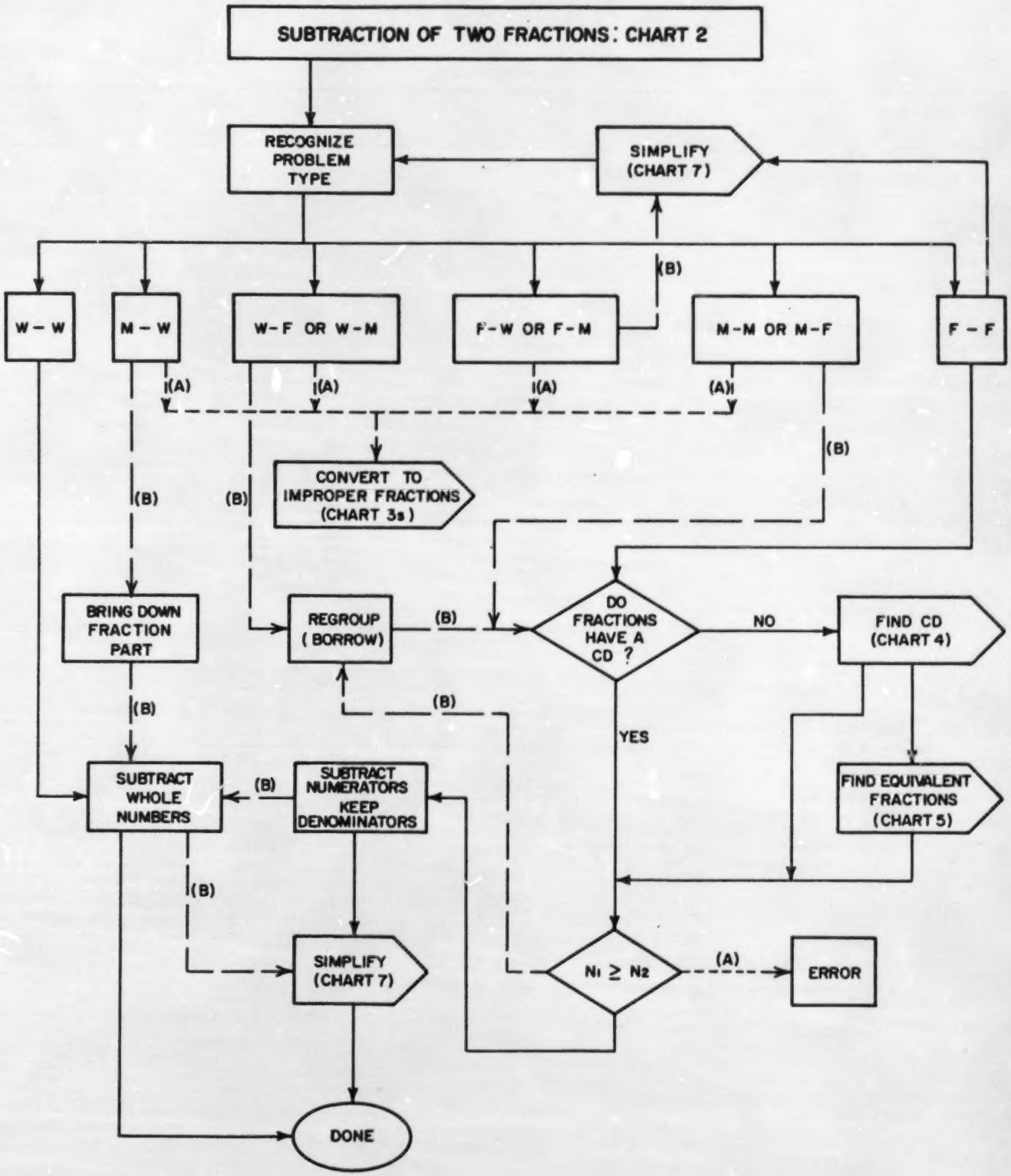

Figure 1B: A Procedural Network for Subtracting Two Fractions

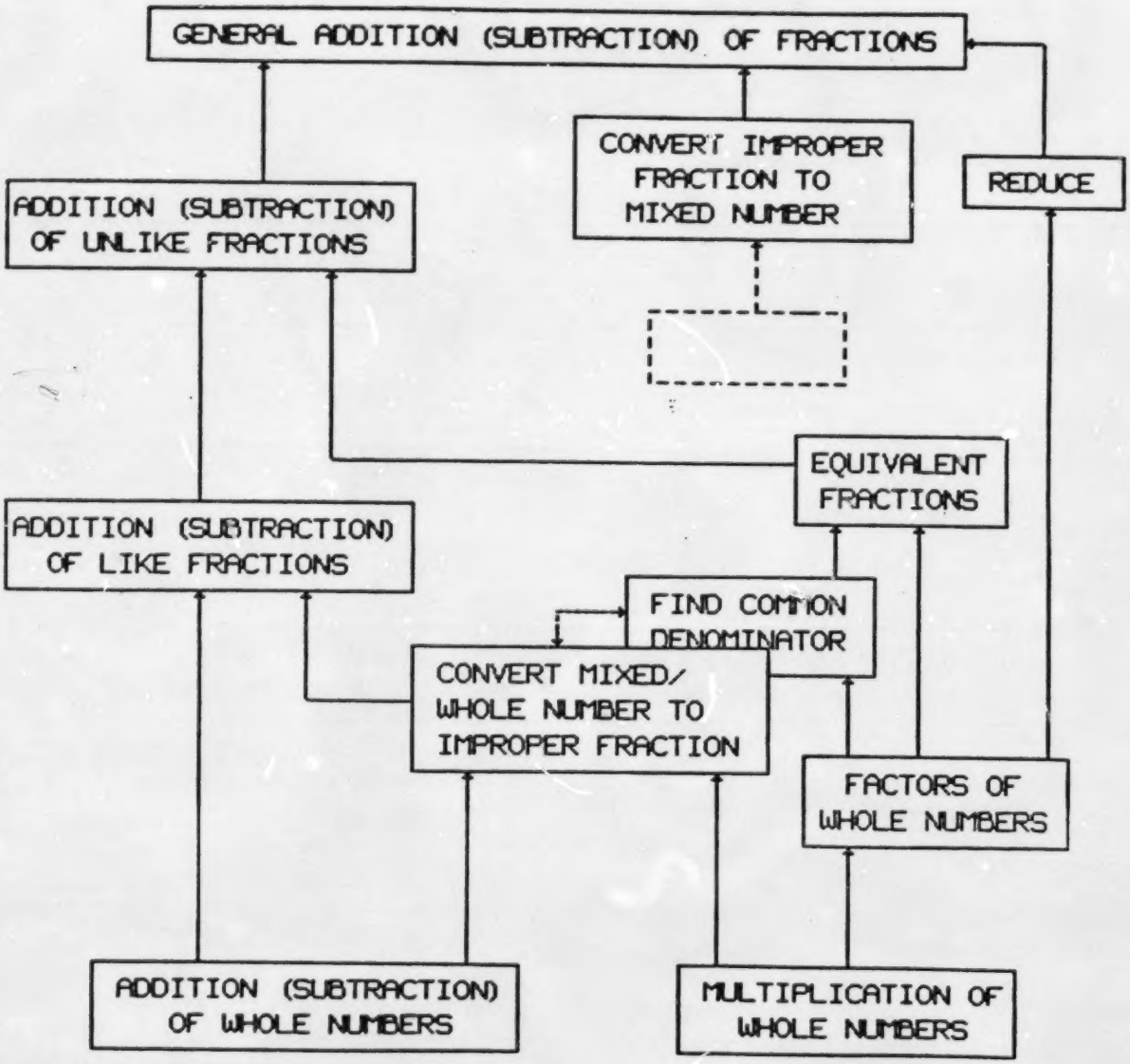
GENERAL ADDITION (SUBTRACTION) OF FRACTIONS
CONVERT IMPROPER FRACTION TO MIXED NUMBER
REDUCE
ADDITION (SUBTRACTION) OF UNLIKE FRACTIONS
EQUIVALENT FRACTIONS
ADDITION (SUBTRACTION) OF LIKE FRACTIONS
FIND COMMON DENOMINATOR
CONVERT MIXED/ WHOLE NUMBER TO IMPROPER FRACTION
FACTORS OF WHOLE NUMBERS
ADDITION (SUBTRACTION) OF WHOLE NUMBERS
MULTIPLICATION OF WHOLE NUMBERS

Figure 2: A Skills Hierarchy for the Addition (Subtraction) of Fractions

Level III: The student is able to add (subtract) fractions (F+F) with unlike denominators. Failure at this level may be due to incorrect algorithms for either finding common denominators or for converting to equivalent fractions.

Level IV: The student is able to generalize the skills for Levels I-III to problem types involving either mixed or whole numbers (F+W, F+M, M+W, M+M). Failure at this level may be due to an incorrect algorithm to convert either mixed or whole numbers to improper fractions.

Level V: The student is able to simplify the sum (difference) by converting an improper fraction to a mixed number and/or reducing the proper fraction part to lowest terms.

Level VI: The student is able to solve problems involving multiple procedures (listed above).

Level VII: The student selects and applies efficient procedures. The strategy is appropriate for the problem a general solution to an addition (subtraction) problem type.

Strategies for Solving Problems Involving
the Addition and Subtraction of Fractions

Strategies for solving fraction addition and subtraction problems can be classified according to the type of problem: fraction-fraction combination or mixed/whole number combination. Unless otherwise noted, each of the strategies is embedded within the procedural network described earlier.

Fraction-fraction: Several F+F strategies have been identified.

1. The BASIC F+F strategy is to apply the simplest correct algorithm to the problem:

$a/b + c/b = (a+c)/b$ (like fractions)

$a/b + c/nb = na/nb + c/nb = (na+c)/nb$ (multiple)

$a/b + c/d = ad/bc + bc/bd = (ad+bc)/bd$ (unlike)

The answer can be simplified either by converting improper fractions to mixed numbers and then reducing or vice-versa. For example,

$7/8 + 5/8 = 12/8 = 3/2 = 1\ 1/2$

$= 1\ 4/8 = 1\ 1/2$

The order of simplifying is probably influenced by the student's ability to factor or divide. For example, a student who is uncomfortable with division by two digit divisors might choose to reduce the answer 54/15 before trying to convert it to a mixed number.

2. The COUNTING strategy is to mentally rename the numbers in a form that eliminates converting from an improper fraction to a mixed number. For example,

4/5 + 3/5 → (mental work) 4/5 + 1/5 + 2/5 =

1 + 2/5 = 1 2/5

We probably cannot determine whether or not a student uses this method without asking. Since application of this method is not transparent, we have not included it in the procedural network.

3. The AUTOMATIC F+F strategy is one in which the student uses the general formula

a/b + c/d = (ad+bc)/bd for every problem type by substituting values. The student does not need to differentiate between problems involving like and unlike fractions.

4. SIMPLIFICATION first: In this case the student simplifies the fractions in the problem before adding (subtracting). This strategy is not often stressed by teachers since few texts include appropriate problems.

Mixed/Whole Number Combinations: In this section we describe two algorithms for solving problems that contain either mixed or whole numbers.

1. Method A: Each mixed or whole number is converted to an improper fraction and then an F+F strategy is used to add (subtract) the fractions. For example,

3 1/5 + 1 3/5 = 16/5 + 8/5 = 24/5 = 4 4/5

One advantage of Method A is that borrowing is never needed. As a result, signed number fraction arithmetic is simplified. One disadvantage is that students work with larger numbers which might make reducing and converting to mixed numbers more difficult. For example,

1 1/8 + 2 1/6 → 9/8 + 13/6

→ 27/24 + 52/24

→ 79/24 = 3 7/24

2. Method B: The student adds (subtracts) the fraction parts separately by using an F+F strategy. The whole number parts are then added. Finally, the two separate answers are combined and simplified. For example,

3 1/5 → 3 + 1/5

1 3/5 → 1 + 3/5

4 + 4/5 = 4 4/5

One advantage is that the student is manipulating relatively small numbers. The major disadvantage is shown in subtraction problems in which borrowing is necessary.

Procedure of Item Construction

One of the most common errors in getting the answer to an F+F type fraction problem is to multiply the two denominators and add the numerators. For example, suppose a student writes his/her answer as 1 for the item 1/2 + 3/2 on a test paper. From this answer alone it is difficult to know how the student processed the problem. There are two possible rules--multiply two denominators or add them. In order to find which is the student's rule, a second item is needed, say 8/5 + 6/5. If the student writes 14/25 on the sheet then he multiplied them. Another misconception closely related to this operation is that the student applies the operation only when the denominators are the same. If this is the case, then these two items are not enough to diagnose the student's rule. A third item, say of the type 3/5 + 13/3, must be added to the test. If the student's answer is 16/15, then the rule will be diagnosed. But if the student also has a misconception involving the operation of converting improper fractions to a mixed number, then the answer will be a different number.

Suppose the student's answer is 1 1/15 despite his having the misconception of putting his answer N W/D while the right answer should be W N/D. Then there is no way to judge whether the student still has a wrong idea of conversion. We must give him another type of item which can clarify and eliminate all other candidates of his/her erroneous rules. Since the value of the denominator in his/her answer is 15, he

multiplies the two denominators when they are unlike. We have to search for a fourth item for this student so as to determine his/her rule. The item, 4/5 + 13/3, will give the solution we have sought unless the student has a further combination of misconceptions.

Similar procedures were applied consistently while our two tests were being constructed. It will be too lengthy to describe the history of these painstaking procedures which demand an enormous amount of concentrated attention. It is urgent to develop more efficient and simpler procedures of test construction for this purpose.

Summary and Discussion

The problem types of addition and subtraction problems are classified into several categories and logical analysis of step-by step procedures for completing each task in the category was described and represented in graphical networks. A number of projected erroneous rules is listed in the Appendix. A variety of different error types and a fairly large number of erroneous rules demonstrate a complexity of human cognition and suggest the difficulty of efficient teaching. Providing specific descriptions of misconceptions will be useful in understanding why a student cannot master fraction arithmetic, but the problem of how to utilize these specific prescrptions of errors in designing efficient remedial instruction remains unsolved. The problem of how to sort and deal with hundreds and thousands of "bugs" remains unsolved. New psychometric models by which "bugs" will be classified into several categories -- persistent errors, robust errors, easty-to-remove errors, etc.-- must be developed.

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Appendix I

Contracted Version of the 48-Item Fraction Addition Test

1. $2\frac{8}{6} + 3\frac{10}{6} =$
2. $\frac{2}{5} + \frac{12}{8} =$
3. $\frac{8}{5} + \frac{6}{5} =$
4. $2\frac{1}{2} + 4\frac{2}{4} =$
5. $\frac{1}{2} + 1\frac{10}{7} =$
6. $3\frac{5}{2} + 4\frac{6}{7} =$
7. $\frac{3}{5} + \frac{7}{5} =$
8. $\frac{1}{3} + \frac{1}{2} =$
9. $1\frac{4}{7} + 1\frac{12}{7} =$
10. $\frac{3}{5} + \frac{1}{5} =$
11. $\frac{3}{4} + \frac{1}{2} =$
12. $2\frac{5}{9} + 1\frac{1}{9} =$
13. $3\frac{1}{6} + 2\frac{3}{4} =$
14. $\frac{15}{35} + \frac{10}{35} =$
15. $\frac{1}{2} + \frac{3}{8} =$
16. $1\frac{2}{5} + \frac{3}{5} =$
17. $\frac{1}{4} + \frac{3}{4} =$
18. $\frac{4}{15} + \frac{1}{10} =$
19. $\frac{4}{5} + \frac{3}{5} =$
20. $3 + \frac{10}{5} =$
21. $1\frac{3}{9} + 2 =$
22. $3 + 1\frac{5}{2} =$
23. $\frac{12}{8} + 1$
24. $2\frac{1}{3} + 3\frac{1}{4} + 2\frac{2}{6} =$
25. $3\frac{10}{4} + 4\frac{6}{4} =$
26. $\frac{2}{7} + \frac{18}{12} =$
27. $\frac{9}{7} + \frac{11}{7} =$
28. $1\frac{1}{3} + 2\frac{4}{6} =$
29. $\frac{1}{5} + 2\frac{5}{3} =$
30. $3\frac{4}{5} + 5\frac{3}{5} =$
31. $\frac{7}{4} + \frac{5}{4} =$
32. $\frac{1}{5} + \frac{1}{4} =$
33. $1\frac{3}{5} + 1\frac{8}{5} =$
34. $\frac{4}{7} + \frac{1}{7} =$
35. $\frac{5}{6} + \frac{1}{3} =$
36. $3\frac{5}{8} + 1\frac{1}{8} =$
37. $2\frac{1}{8} + 3\frac{5}{6} =$
38. $\frac{16}{36} + \frac{10}{36} =$
39. $\frac{1}{3} + \frac{4}{9} =$
40. $2\frac{5}{7} + \frac{2}{7} =$
41. $\frac{1}{5} + \frac{4}{5} =$
42. $\frac{5}{6} + \frac{1}{8} =$
43. $\frac{6}{7} + \frac{3}{7} =$
44. $5 + \frac{8}{4} =$
45. $1\frac{2}{6} + 2 =$
46. $2 + 1\frac{7}{3} =$
47. $\frac{18}{12} + 3 =$
48. $2\frac{1}{3} + 1\frac{2}{9} + 3\frac{1}{2} =$

Appendix II

Contracted Version of the 48-Item Fraction Subtraction Test

1. $\frac{5}{3} - \frac{3}{4} =$
2. $\frac{3}{4} - \frac{3}{8} =$
3. $\frac{5}{6} - \frac{1}{9} =$
4. $3\frac{1}{2} - 2\frac{3}{2} =$
5. $4\frac{3}{5} - 3\frac{4}{10} =$
6. $\frac{6}{7} - \frac{4}{7} =$
7. $3 - 2\frac{1}{5} =$
8. $\frac{2}{3} - \frac{2}{3} =$
9. $3\frac{7}{8} - 2 =$
10. $4\frac{4}{12} - 2\frac{7}{12} =$
11. $4\frac{1}{3} - 2\frac{4}{3} =$
12. $\frac{11}{8} - \frac{1}{8} =$
13. $3\frac{3}{8} - 2\frac{5}{6} =$
14. $3\frac{4}{5} - 3\frac{2}{5} =$
15. $2 - \frac{1}{3} =$
16. $4\frac{5}{7} - 1\frac{4}{7} =$
17. $7\frac{3}{5} - \frac{4}{5} =$
18. $4\frac{1}{10} - 2\frac{8}{10} =$
19. $4 - 1\frac{4}{3} =$
20. $4\frac{1}{3} - 1\frac{5}{3} =$
21. $5 - (7\frac{1}{3} - 2\frac{1}{3}) =$
22. $\frac{8}{5} - \frac{5}{6} =$
23. $\frac{5}{3} - \frac{5}{6} =$
24. $\frac{5}{6} - \frac{1}{15} =$
25. $4\frac{1}{3} - 3\frac{4}{3} =$
26. $3\frac{2}{3} - 2\frac{2}{6} =$
27. $\frac{3}{4} - \frac{2}{4} =$
28. $4 - 3\frac{1}{6} =$
29. $\frac{3}{4} - \frac{3}{4} =$
30. $4\frac{8}{9} - 2 =$
31. $5\frac{3}{15} - 3\frac{8}{15} =$
32. $5\frac{1}{4} - 3\frac{5}{4} =$
33. $\frac{13}{9} - \frac{1}{9} =$
34. $4\frac{4}{9} - 3\frac{5}{6} =$
35. $4\frac{5}{7} - 4\frac{3}{7} =$
36. $2 - \frac{1}{4} =$
37. $5\frac{7}{9} - 1\frac{5}{9} =$
38. $8\frac{1}{3} - \frac{2}{3} =$
39. $5\frac{1}{10} - 3\frac{4}{10} =$
40. $5 - 2\frac{5}{4} =$
41. $5\frac{1}{5} - 2\frac{7}{5} =$
42. $4 - (6\frac{1}{4} - 2\frac{1}{4}) =$

Appendix III

Projected Errors For Addition and Subtraction of Fractions

Addition Errors: Inappropriate and Incorrect Algorithms.

A1. The student uses the following algorithm:

a. If the numerators are not equal, add them; otherwise, keep the same numerator.

b. If the denominators are not equal, add them; otherwise keep the same denominator.

c. If the whole numbers are not equal, add them; otherwise keep the the same whole number.

A2. The components of the answer are calculated as follows:

a. The numerator is the sum of the numerators.

b. The denominator is the sum of the denominators.

c. The whole number part is the sum of the whole numbers.

A3. The student uses the correct algorithm for division.

A4. Like A3, but the student inverts the first fraction.

A5. Like A3, but the student inverts both fractions.

A6. The components of the answer are calculated as follows:

a. The numerator is the sum of the denominators

b. The denominator is the product of the denominators.

c. The whole number part is the sum of the whole numbers.

A7. The components of the answer are calculated as follows:

a. The numerator is the sum of the numerators.

b. The denominator is the larger of the denominators

c. The whole number part is the sum of the whole numbers.

A8. The components of the answer are calculated as follows:

a. The numerator is the product of the numerators.

b. The denominator is the product of the denominators.

c. The whole number part is the product of the whole numbers.

A10. The components of the answer are calculated as follows:

a. The numerator is the sum of the numerators.

b. The denominator is the product of the denominators.

c. The whole number part is the sum of the whole numbers.

A11. The components of the answer are calculated as follows:

a. The numerator is the product of the numerators.

b. The denominator is the larger of the two denominators.

c. The whole number part is the sum of the whole numbers.

A12. The components of the answer are calculated as follows:

a. The numerator is correct.

b. The denominator is correct.

c. The student fails to add the whole number parts.

A13. Like A11, but the error occurs only with problems M+F or F+M.

A14. Like A12, but the error occurs only with the case F+M but not with M+F.

A15. The student fails to add the whole number parts in W+F or F+W.

A16. The student "cross cancels" wherever he can. Is more frequent when problems are written vertically.

$$\frac{\not{2}}{3} + \frac{1}{\not{2}} = \frac{1}{3}$$

A17. Like A12 but the student loses the whole number parts only in problems involving finding a common denominator.

A18. In cases such as W+F, the student inserts a fraction equal to one next to the whole number (Method B only). For example:

$3 + 2/5 \rightarrow 3\ 5/5 + 2/5 = 3\ 7/5 = 4\ 2/5$

A19. In cases such as W+F, the student borrows one from W. The final answer is correct if no other errors occur. The procedure is not incorrect; just inefficient.

A20. The student uses a correct subtraction algorithm.

A21. The student uses an incorrect subtraction algorithm.

Subtraction Errors: Inappropriate and Incorrect Algorithms.

S1. The student uses the following algorithm:

a. If the numerators are not equal, subtract and take the absolute value; otherwise, keep the same numerator.

b. If the denominators are not equal, subtract and take the absolute value; otherwise keep the same denominator.

c. If the whole numbers are not equal, subtract and take the absolute value; otherwise keep the same whole number.

S2. The components of the answer are calculated as follows:

a. The numerator is the difference of the absolute values of of the numerators.

b. The denominator is the difference of the absolute values of the denominators.

c. The whole number part is the difference of the absolute value of the whole numbers.

S3. The student uses the correct algorithm for division.

S4. Like S3, but the student inverts the first fraction.

S5. Like S3, but the student inverts both fractions.

S6. The components of the answer are calculated as follows:

a. The numerator is the difference of the absolute values of the denominators

b. The denominator is the product of the denominators.

c. The whole number part is the difference of the absolute values of the whole numbers.

S7. The components of the answer are calculated as follows:

a. The numerator is the difference of the absolute values of the numerators.

b. The denominator is the larger of the denominators

c. The whole number part is the difference of the absolute values of the whole numbers.

S8. The student uses the correct algorithm for multiplication.

S9. The components of the answer are calculated as follows:

a. The numerator is the product of the numerators.

b. The denominator is the product of the denominators.

c. The whole number part is the product of the whole numbers.

S10. The components of the answer are calculated as follows:

a. The numerator is the difference of the absolute values of the numerators.

b. The denominator is the product of the denominators.

c. The whole number part is the difference of the absolute values of the whole numbers.

S11. The components of the answer are calculated as follows:

a. The numerator is correct.

b. The denominator is correct.

c. The student fails to subtract the whole number parts.

S12. Like S11 but the error occurs only with problems M-F or F-M.

S13. Like S11 but the error occurs only with the case F-M but not with M-F.

S14. The student fails to add the whole number parts in W-F or F-W.

S15. The student "cross cancels" wherever he can. Is more frequent when problems are written vertically.

$$\begin{array}{r} \overset{2}{\cancel{2}} \\ 3 \\ +\ \overset{1}{\cancel{2}} \\ \hline \end{array} \quad \begin{array}{r} 1 \\ 3 \\ \hline \end{array}$$

S16. Like S11 but the student loses the whole number parts only in problems involving finding a common denominator.

S17. The student uses a correct addition algorithm.

S18. The Student does not answer any question of a particular type. The student skips problems:

a. W-F

b. F-F

c. M-F

d. W-M

e. M-M

f. F-M

g. M-W

h. F-W

i. involving borrowing

i. involving finding a common denominator

Subtraction errors involving borrowing

Unless otherwise indicated, these errors occur with problem types W-F, W-M, M-F, and M-M.

B1. The student subtracts only the whole number parts of problem types W-F and W-M.

B2. When the student borrows 1 from the whole number, he converts it to a 10 and adds it to the numerator of the fraction.

B3. When the student borrows from the whole number he does not change the whole number.

B4. When the student borrows from the whole number, he drops the fraction part of the mixed number.

B5. After finding the common denominator, the student uses the following algorithm:

a. the numerator is found by subtracting the smaller numerator from the larger

b. the denominator is the common denominator

c. the whole number is the difference in the absolute values of the whole numbers.

B6. The student adds 1 to the whole number instead of subtracting 1 when borrowing.

B7. In problem types M-W and F-W the student converts W to a mixed fraction by borrowing 1 from W. This forces the student to borrow one more time. Also, the increased number of steps increases the likelihood of errors.

B8. This error occurs in converting a mixed number to an improper fraction. The student uses Method B but borrows 1 and changes the fraction part to a mixed number with 1 as its whole number part.

Errors in finding equivalent fractions

E1. The student uses the following algorithm to find an equivalent fraction:

a. The numerator is the product of the numerator and the denominator.

b. The denominator is the product of the two denominators.

E2. The student uses the following algorithm to find an equivalent fraction:

a. The numerator is the denominator of the original fraction.

b. The denominator is the product of the two denominators.

E3. The student uses the following algorithm to find an equivalent fraction:

a. The numerator is the quotient of the common denominator divided by the original denominator

b. The denominator is the product of the two denominators.

E4. Like E1 except that the denominator is the least common denominator.

a. The numerator is the product of the numerator and the denominator.

b. The denominator is the product of the two denominators.

E5. The student uses the following algorithm to find an equivalent fraction:

a. The numerator is the product of the original numerator and the denominator of the other fraction.

b. The denominator is the product of the two numerators.

E6. The student changes whole numbers to fractions by writing the whole number as the numerator and denominator of the equivalent fraction.

E7. In problem types M+M and M+F, the student loses the whole number portion after converting to equivalent fractions. He fails to add the whole number parts.

Inefficient algorithms related to finding the common denominator:

CD1. Student uses a common denominator but not the lowest common denominator (LCD).

CD2. Student uses the product of denominators as the common denominator in all problems.

Errors in converting to improper fraction:

IF1. The numerator of the equivalent fraction is calculated using one of these errors:

a. The student multiplies the denominator, whole number, and numerator.

b. The student adds the denominator, whole number, and numerator.

c. The student multiples the numerator times the sum of the denominator and whole number

d. The student multiplies the whole number and the numerator and then adds the denominator.

e. The student multiplies the denominator and the whole number but ignores the numerator.

f. The student multiples the numerator and the whole number.

g. The student adds the whole number and the denominator.

h. The student multiplies the whole number and the denominator.

IF2. The student changes whole numbers to their reciprocal.

IF3. The student changes the order of the fractions. This is likely only if the problem were written horizontally.

Simplifying errors involving reducing

R1. The student does not reduce his answer.

R2. The student does not reduce completely.

R3. The student drops the whole number part of the mixed number.

R4. The student gives the

R4. The student gives the reciprocal of the correct answer.

R5. Like R4 but the error occurs only if the correct answer is a whole number.

R6. The student reduces only in certain cases:

a. if the numerator and denominator have a common factor

b. if the numerator and denominator have a common factor of 2.

c. if the numerator and denominator have a common factor of 2 or 3.

d. if the numerator and denominator have a common factor of 2, 3, or 5.

e. if the numerator and denominator are both single digit numerals.

R7. The student cancels the one's digit in the numerator or denominator.

R8. The student cancels any digits that appear in both the numerator and denominator.

R9. The student recognizes that 2 is the common factor but divides each number by its other factor.

R10. When using Method B, the student drops the whole number when reducing.

R11. If the numerator is even, the student divides it by two but keeps the old denominator.

R12. The student divides the numerator by the common factor of the numerator and denominator, but keeps the old denominator.

R13. The student divides the numerator and denominator by different numbers.

Simplifying errors involving converting and reducing

CR1. In problems involving both converting and reducing, the student converts but loses the whole number portion when he reduces.

CR2. When using Method B, the student drops the whole number obtained in adding when he converts or reduces.

CR3. Answers with a 0 in the numerator are given as 0.

CR4. Answers with a 0 in the numerator are not simplified.

CR5. Answers with a 1 in the denominator are not simplified.

CR6. Answers with a 1 in the numerator are given as the reciprocal.

CR7. When converting improper fractions to mixed numbers, the student interchanges the whole number, the numerator, and the denominator

CR8. Like CR6 but the values for the whole number, numerator, and denominator are determined by the relative sizes of the quotient, remainder, and divisor.

CR9. The student converts a proper fraction to a mixed number.

CR10. The student converts only in certain circumstances:

a. The improper fraction is equivalent to 1.

b. The numerator is a multiple of the denominator.

c. The arithmetic is relatively easy.

d. The numerator and denominator have a common factor and the denominator is less than 10.

CR11. Mixed numbers with the fraction portion of the form n/n are converted to 1.

CR12. The student does not convert improper fractions.

CR13. The student drops the whole number part in a mixed number.

CR14. The student uses subtraction in converting from improper fractions to mixed numbers.

CR15. The student loses the whole numbe part when he converts improper mixed numbers to proper mixed fractions but not when he converts an improper fraction to a mixed number.

CR16. When converting from improper fractions to mixed numbers, the student always uses 1 as the whole number part.

Appendix IV

Procedural Network Charts

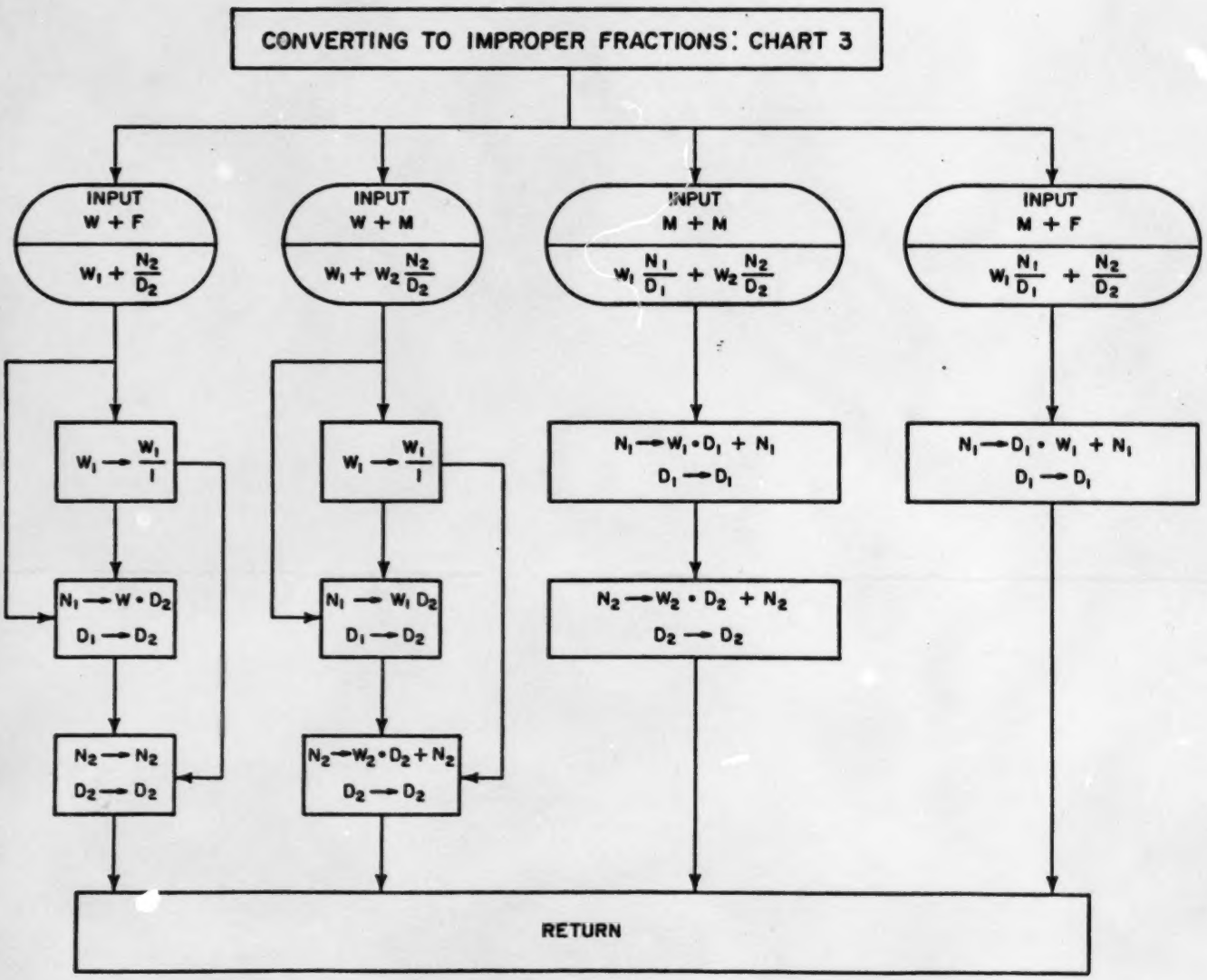

FIGURE 3: A Flow Chart to Convert Whole Numbers or Mixed Fractions to IMPROPER FRACTIONS.

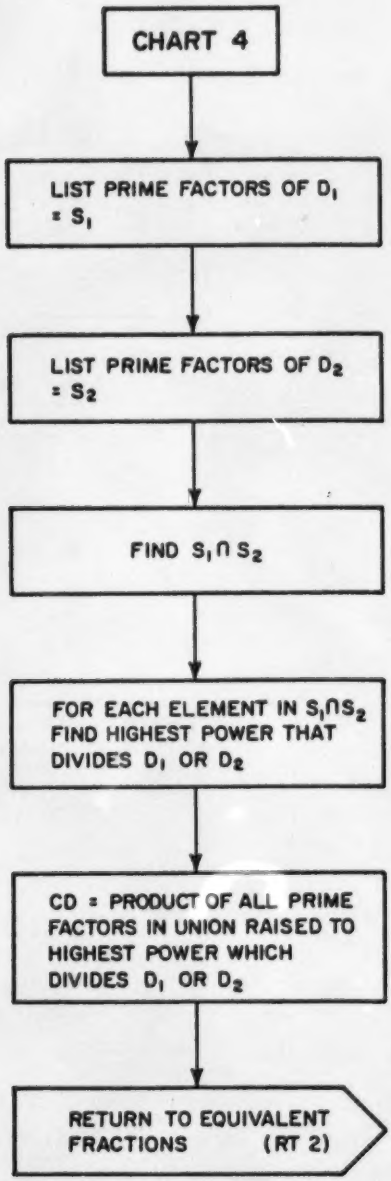
CHART 4
LIST PRIME FACTORS OF D_1 = S_1
LIST PRIME FACTORS OF D_2 = S_2
FIND $S_1 \cap S_2$
FOR EACH ELEMENT IN $S_1 \cap S_2$ FIND HIGHEST POWER THAT DIVIDES D_1 OR D_2
CD = PRODUCT OF ALL PRIME FACTORS IN UNION RAISED TO HIGHEST POWER WHICH DIVIDES D_1 OR D_2
RETURN TO EQUIVALENT FRACTIONS (RT 2)

FIGURE 4: A Flow Chart for Finding Common Denominators
(1) Prime Factoring Method

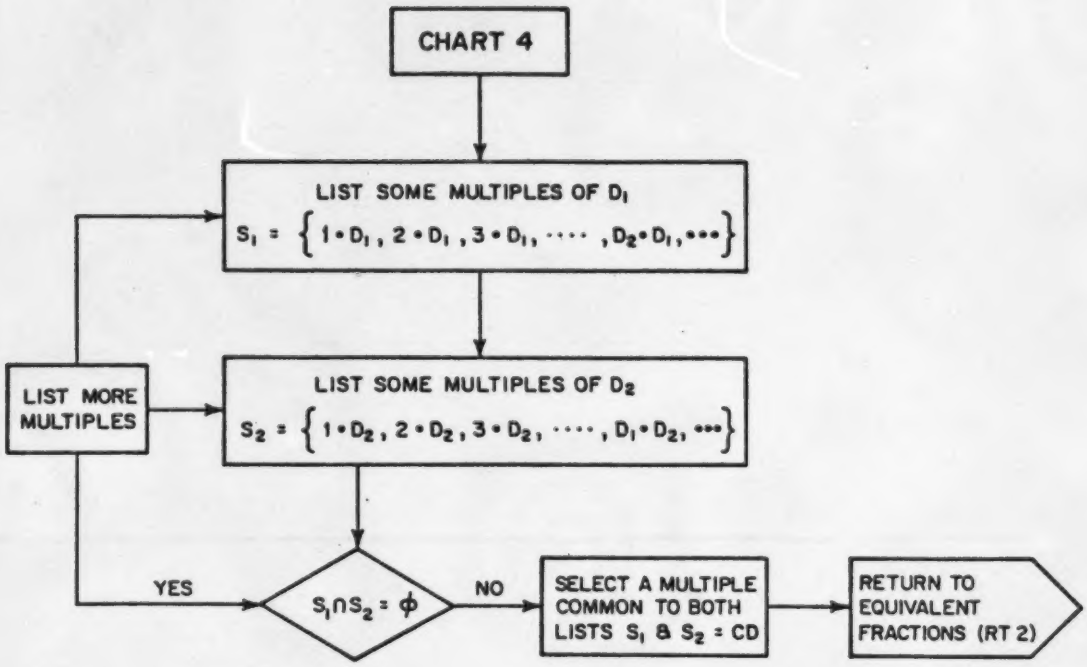
CHART 4
LIST SOME MULTIPLES OF D_1
$S_1 = \{1 \cdot D_1, 2 \cdot D_1, 3 \cdot D_1, \cdots, D_2 \cdot D_1, \cdots\}$
LIST MORE MULTIPLES
LIST SOME MULTIPLES OF D_2
$S_2 = \{1 \cdot D_2, 2 \cdot D_2, 3 \cdot D_2, \cdots, D_1 \cdot D_2, \cdots\}$
YES
$S_1 \cap S_2 = \phi$
NO
SELECT A MULTIPLE COMMON TO BOTH LISTS S_1 & S_2 = CD
RETURN TO EQUIVALENT FRACTIONS (RT 2)

FIGURE 4a : A Flow Chart for Finding Common Denominators
(2) The Multiples Method

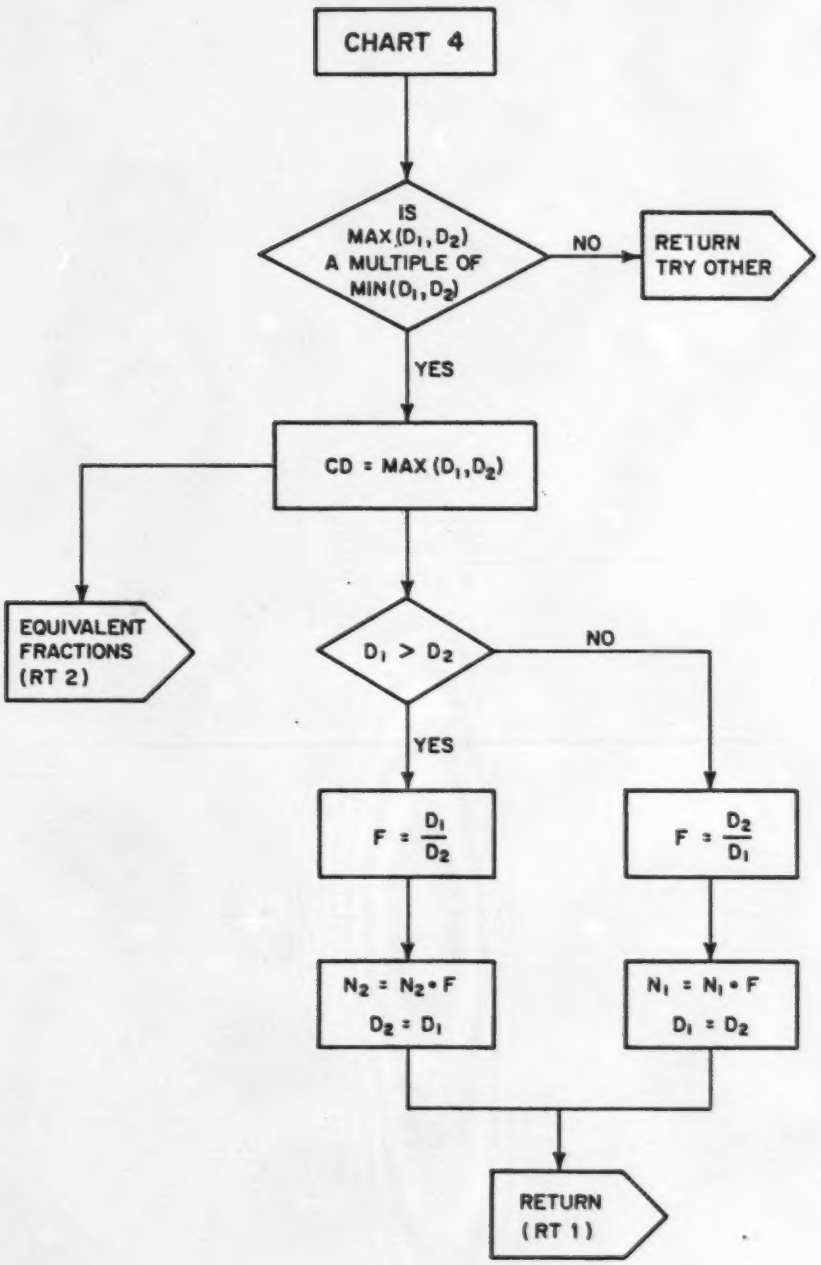

FIGURE 4 b: A Flow Chart for Finding Common Denominators
(3) Either Denominator is a Multiple of the Other

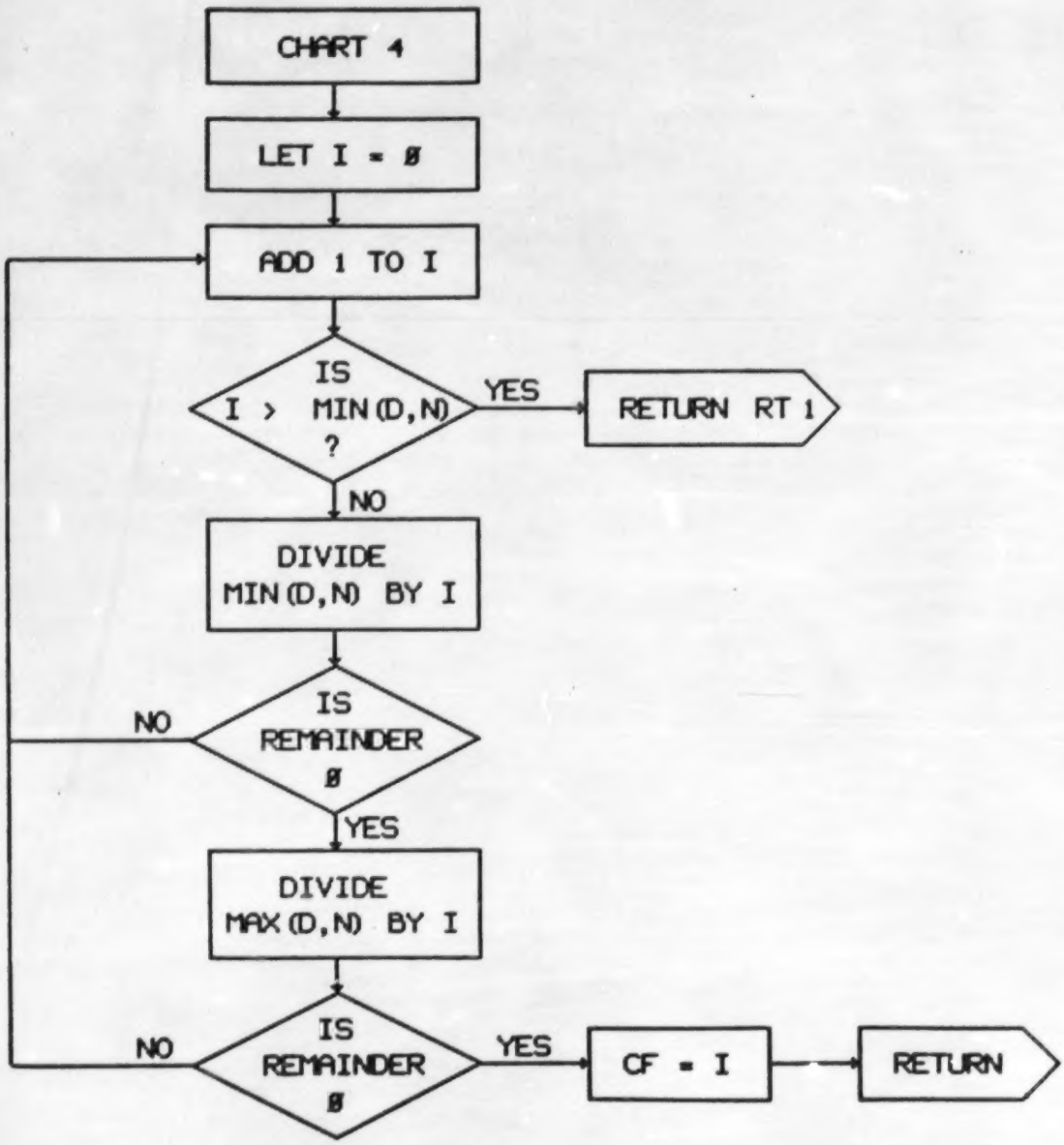

Figure 4c: A Flow Chart for Finding Common Denominators
(4) The Division Method

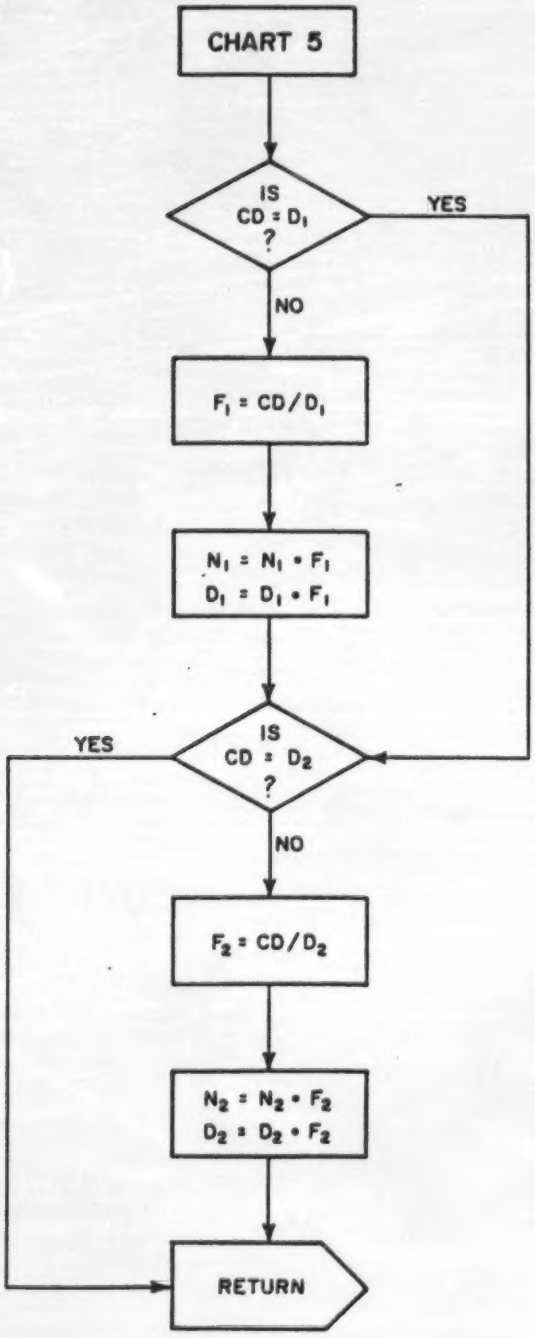

FIGURE 5 : A Flow Chart to Find Equivalent Fractions

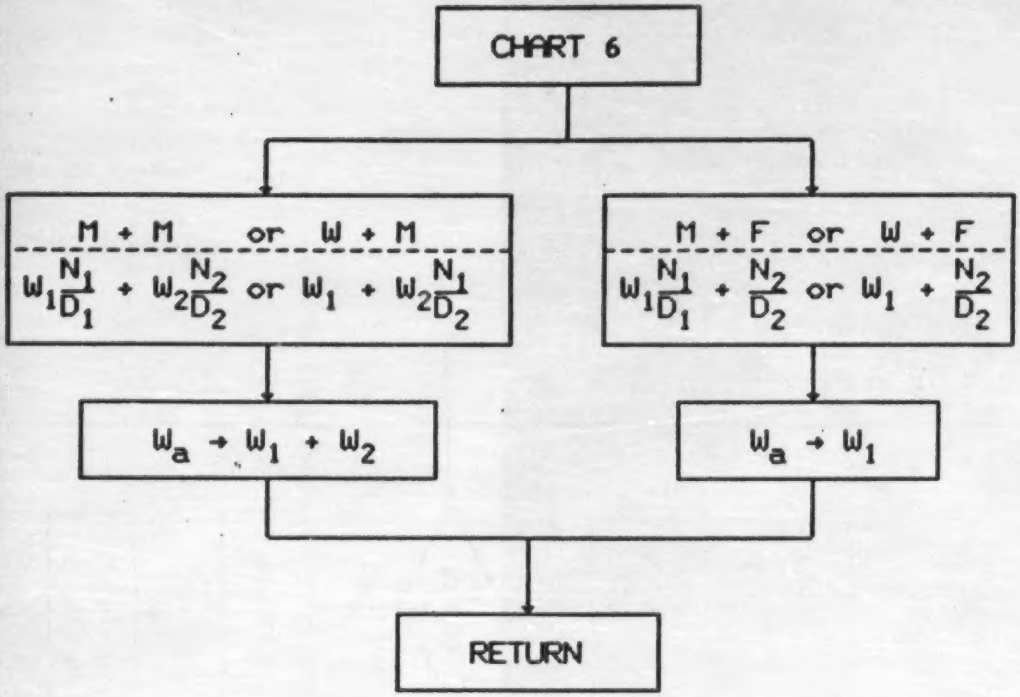
CHART 6
M + M or W + M
$W_1\frac{N_1}{D_1} + W_2\frac{N_2}{D_2}$ or $W_1 + W_2\frac{N_1}{D_2}$
M + F or W + F
$W_1\frac{N_1}{D_1} + \frac{N_2}{D_2}$ or $W_1 + \frac{N_2}{D_2}$
$W_a \rightarrow W_1 + W_2$
$W_a \rightarrow W_1$
RETURN

Figure 6 : A Flow Chart to Add Whole Number Parts

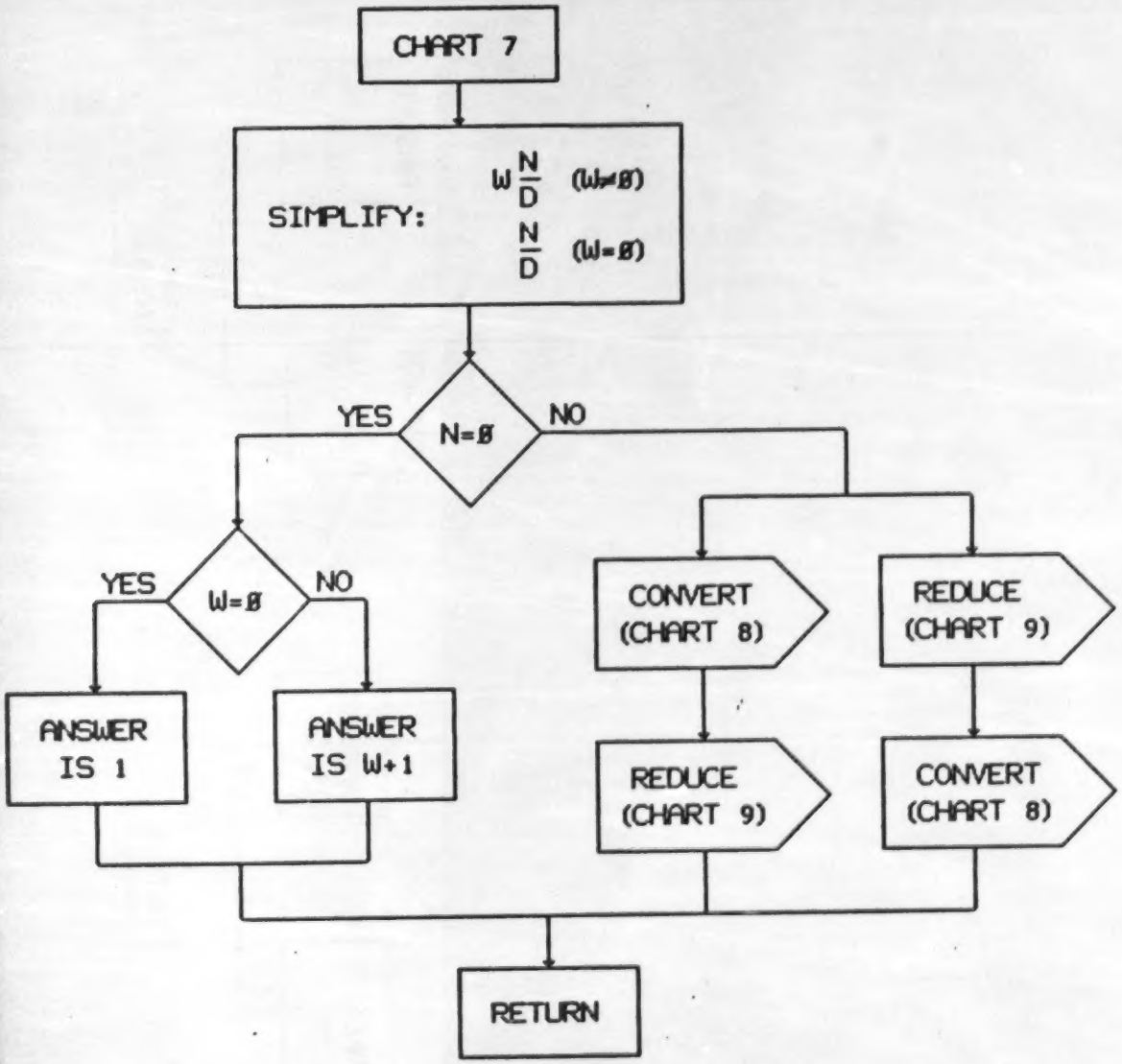

Figure 7: A Flow Chart to Simplify a Fraction

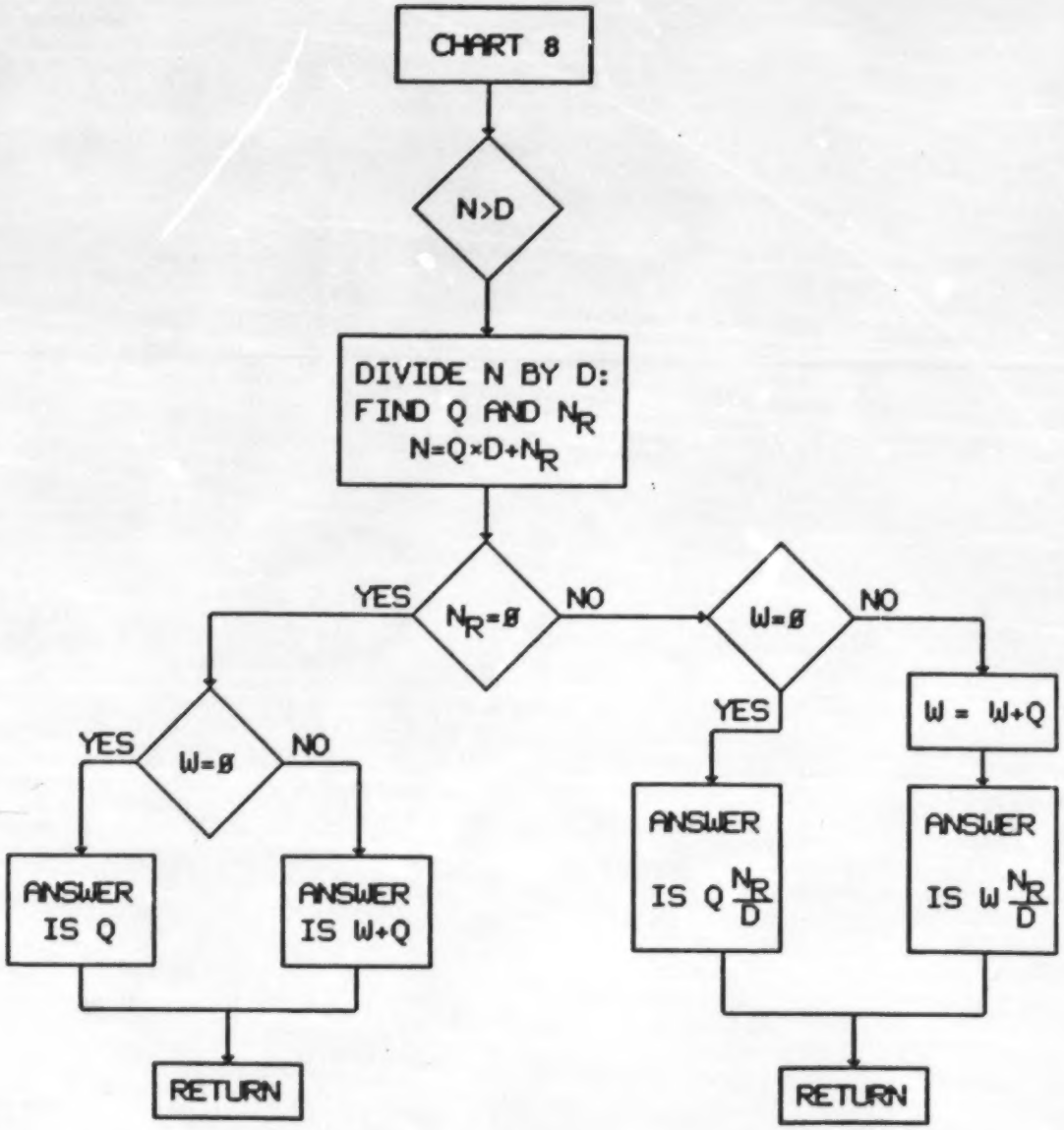

Figure 8: A Flow Chart to Convert an Improper Fraction to a Mixed Number

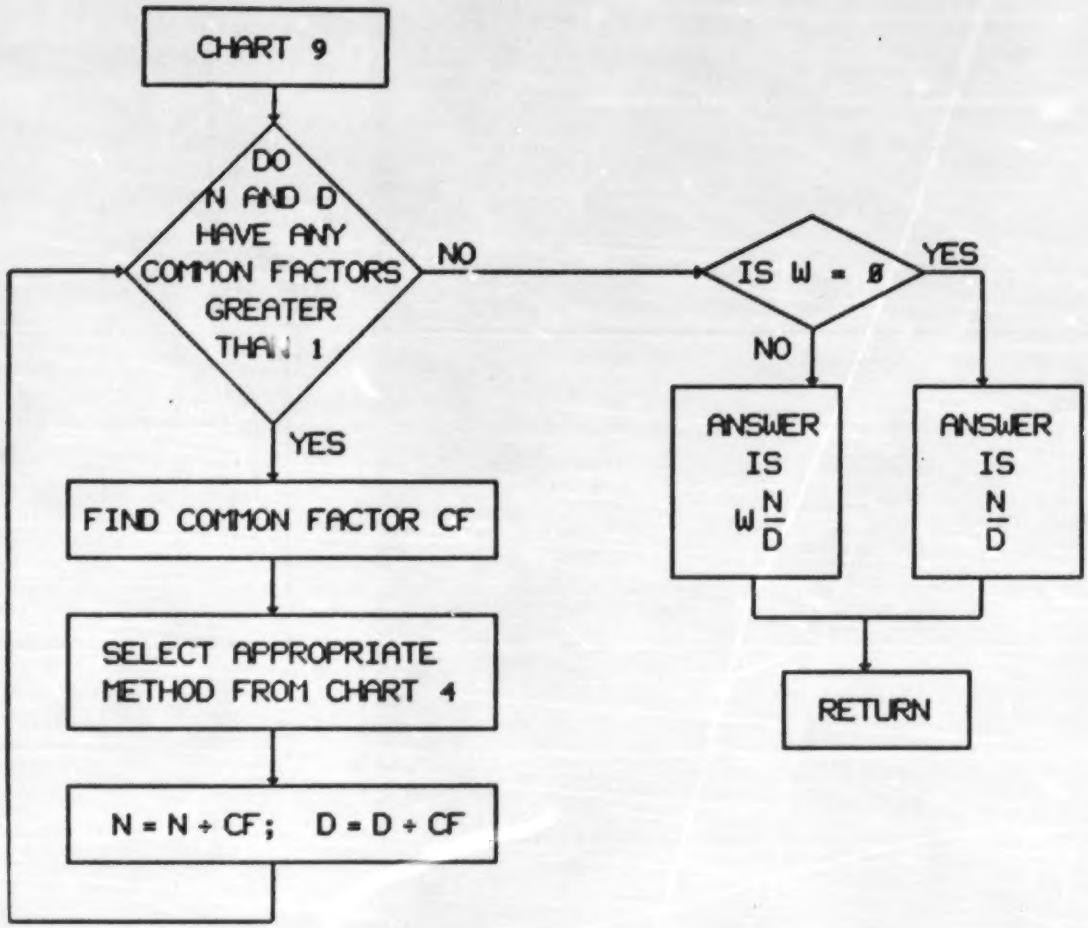

Figure 9: A Flow Chart to Reduce a Fraction

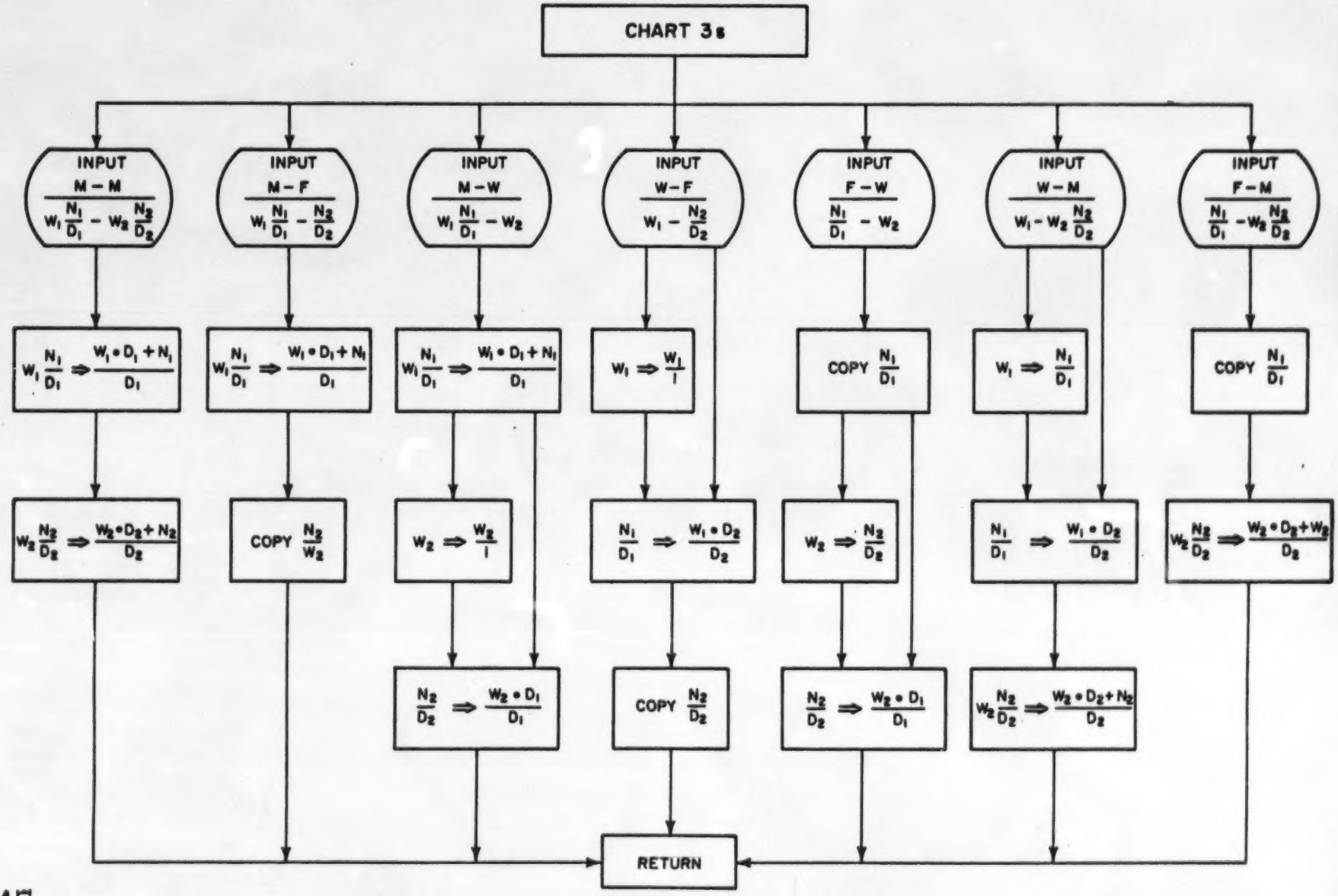

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FIGURE 3s A Flow Chart to Convert Whole Numbers or Mixed Fractions to Improper Fractions During Subtraction

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